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Completeness relations for the electromagnetic modes of a cylindrical fibre with a radially dependent dielectric and magnetic permittivity and conductivity

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Abstract. We consider an infinitely long conducting cylinder whose dielectric and magnetic permittivity and conductivity are functions of the distance from a point inside the cylinder to its axis. It is shown that the r-dependent part of the set of electromagnetic modes associated with such a cylinder is complete and orthogonal, and several completeness relations are constructed, which are different from those postulated in the literature.

1. Introduction

We consider a cylindrical rod (fibre) with radius b whose axis coincides with the z axis of a cylindrical coordinate system, r, ϕ and z. The dielectric permeability ε, the magnetic permeability μ and conductivity σ are supposed to be piecewise once continuously differentiable scalar functions of r.

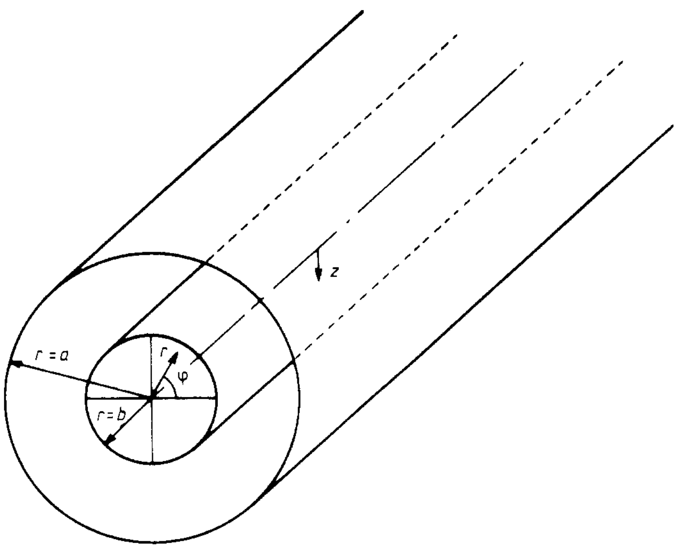

Figure 1. The configuration.

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Connected with this fibre is a set of electromagnetic modes, which are defined as solutions of Maxwell's equations both inside and outside the fibre, satisfying the continuity conditions at the boundary of the fibre.

An interesting problem with many applications is the completeness of this set of modes in a plane perpendicular to the axis of the cylinder. Almost equally important is the construction of explicit expressions for the expansion coefficients once completeness has been shown. We refer with respect to these two problems to Tamir (1979, ch II) who proves the orthogonality relation

$$\iint_{-\infty}^{+\infty} (\boldsymbol{E}_t \times \boldsymbol{H}_t^*)\, \mathrm{d}x\, \mathrm{d}y = \delta(\beta - \beta'), \tag{1.1}$$

for the transverse parts of the electric and magnetic field vectors of the electromagnetic modes of a cylindrical fibre with constant ε, μ and σ equal to zero. The parameter β stems from the ansatz

$$\boldsymbol{E} = \boldsymbol{E}(r, \phi; \beta) \exp(\mathrm{i}\beta z), \qquad \boldsymbol{H} = \boldsymbol{H}(r, \phi; \beta) \exp(\mathrm{i}\beta z). \tag{1.2}$$

The completeness of this set of modes does not follow from Titchmarsh's (1970) book on Sturm-Liouville problems, as erroneously stated by Tamir and Oliner (1963), because Titchmarsh does not analyse Sturm-Liouville theory for systems of coupled differential equations. (See equations (2.15a, b) with $\sigma = 0$, and ε constant.) However, Sturm-Liouville theory for a set of coupled differential equations is well established (Birkhoff 1908, Tamarkin 1927), and orthogonality, as well as completeness, follows immediately from the results obtained by these authors.

Though the problem of this paper could be analysed with the well established Sturm-Liouville theory for coupled differential equations, we prefer to derive the desired completeness and expansion relations by a technique which is closely related to the function theoretical methods used by Birkhoff (1908) and Titchmarsh (1970, ch I). By doing so we are able to obtain a class of possible expressions for the expansion coefficients, which might be very useful for applications. For example, the analysis of scattering and transmission problems generated by a cylindrical fibre heavily depends on the possibility to obtain explicit expressions for the expansion coefficients and their approximate evaluation (Tamir 1979, ch II). If, however, we have the possibility to choose from a class of different expressions for the expansion coefficients, we can choose the most convenient expression. This possibility might lead to an improvement of the approximations used for a particular scattering or diffraction problem.

It is, moreover, interesting to observe that the completeness and orthogonality relations derived in this paper (equation (2.37)) differ from the usually postulated one, namely equation (1.1)!

2. Calculational procedure

Maxwell's equations, together with the material equations, read as

$$\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t, \qquad \nabla \times \boldsymbol{H} = \boldsymbol{j} + \partial \boldsymbol{D}/\partial t, \qquad \nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \cdot \boldsymbol{D} = \rho, \tag{2.1a, b, c, d}$$

$$\boldsymbol{j} = \sigma(r)\boldsymbol{E}, \qquad \boldsymbol{B} = \mu(r)\boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon(r)\boldsymbol{E}. \tag{2.1e, f, g}$$

The functions ε, μ and σ depend explicitly on the distance r, and are supposed to be piecewise once continuously differentiable functions of r in the interval $0 \leq r \leq \infty$, which are constant for values of $r \geq b$ (b denotes the radius of the fibre):

$$\varepsilon(r) = \varepsilon_0, \qquad \sigma(r) = \sigma_0, \qquad r \geq b, \\ \mu(r) = \mu_0. \tag{2.2}$$

From (2.1a, b, e, f, g) we derive a vector differential equation for $\boldsymbol{E}$:

$$\nabla \times \left(\frac{1}{\mu(r)} \nabla \times \boldsymbol{E} \right) = -\frac{\partial}{\partial t}\left(\sigma(r)\boldsymbol{E} + \varepsilon(r)\frac{\partial \boldsymbol{E}}{\partial t} \right). \tag{2.3}$$

We will also need the relation

$$\nabla \cdot [\sigma(r)\boldsymbol{E} + \varepsilon(r)\partial \boldsymbol{E}/\partial t] = 0, \tag{2.4}$$

which is obtained by taking the divergence of equation (2.1b) and inserting (2.1e, g). We will consider electromagnetic fields such that

$$\boldsymbol{E} = \boldsymbol{E}(r;\beta)\exp(-\mathrm{i}\omega t + \mathrm{i}l\phi + \mathrm{i}\beta z), \tag{2.5a}$$

$$\boldsymbol{H} = \boldsymbol{H}(r;\beta)\exp(-\mathrm{i}\omega t + \mathrm{i}l\phi + \mathrm{i}\beta z), \qquad l = 0, \pm 1, \pm 2, \ldots. \tag{2.5b}$$

The frequency ω may become a complex number, which means that, with the appropriate choice for the imaginary part of ω, time decaying fields can be considered. The number β can take any complex value and will play the role of the propagation parameter.

Equation (2.3) can be written as

$$\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} + \nabla(\ln \mu) \times (\nabla \times \boldsymbol{E}) = \mathrm{i}\omega\mu\sigma\boldsymbol{E} - \omega^2\mu\varepsilon\boldsymbol{E}. \tag{2.6}$$

The proper interpretation of the operator $\nabla^2 \boldsymbol{E}$ in an arbitrary orthogonal coordinate system is e.g. given by Morse and Feshbach (1953) (table finishing ch I), and reads for a cylindrical coordinate system as

$$\nabla^2 \boldsymbol{E} = \boldsymbol{i}_r\left(\nabla^2 E_r - \frac{E_r}{r^2} - \frac{2}{r^2}\frac{\partial}{\partial\phi}E_\phi \right) + \boldsymbol{i}_\phi\left(\nabla^2 E_\phi - \frac{E_\phi}{r^2} + \frac{2}{r^2}\frac{\partial E_r}{\partial \phi} \right) + \boldsymbol{i}_z \nabla^2 E_z. \tag{2.7}$$

The relation (2.4) reads as

$$\frac{1}{r}\frac{\partial}{\partial r}[r(\sigma - \mathrm{i}\omega\varepsilon)E_r] + \frac{1}{r}\frac{\partial}{\partial\phi}[(\sigma - \mathrm{i}\omega\varepsilon)E_\phi] + \frac{\partial}{\partial z}[(\sigma - \mathrm{i}\omega\varepsilon)E_z] = 0. \tag{2.8}$$

The r and z components of equation (2.6) are

$$\begin{aligned} &\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial}{\partial\phi}(E_\phi) + \frac{\partial}{\partial z}(E_z) \right) \\ &\quad - \left\{ \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \right] E_r - \frac{E_r}{r^2} - \frac{2}{r^2}\frac{\partial}{\partial\phi}(E_\phi) \right\} \\ &\quad = \mathrm{i}\omega\mu\sigma E_r - \omega^2\varepsilon\mu E_r, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial}{\partial\phi}(E_\phi) + \frac{\partial}{\partial z}(E_z) \right) - \left\{ \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \right] E_z \right\} \\ &\quad + \frac{\partial}{\partial r}(\ln\mu)\left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \right) \\ &\quad = \mathrm{i}\omega\mu\sigma E_z - \omega^2\varepsilon\mu E_z. \end{aligned} \tag{2.10}$$

Eliminating the ϕ component E_ϕ of the electric field using (2.8), and inserting (2.5a), leads to

$$\frac{\partial^2 E_r}{\partial r^2}+\frac{3}{r}\frac{\partial E_r}{\partial r}-\beta^2 E_r-\frac{l^2+2}{r^2}E_r+\frac{2}{r}E_r\frac{\partial}{\partial r}[\ln(\sigma-\mathrm{i}\omega\varepsilon)]$$
$$+\frac{2\mathrm{i}\beta}{r}E_z+\frac{\partial}{\partial r}\left(E_r\frac{\partial}{\partial r}[\ln(\sigma-\mathrm{i}\omega\varepsilon)]\right)$$
$$=(-\omega^2\varepsilon\mu+\mathrm{i}\omega\mu\sigma)E_r, \tag{2.11}$$

$$\frac{\partial^2 E_z}{\partial r^2}+\frac{1}{r}\frac{\partial E_z}{\partial r}-\beta^2 E_z-\frac{l^2}{r^2}E_z+\mathrm{i}\beta E_r\frac{\partial}{\partial r}[\ln(\sigma-\mathrm{i}\omega\varepsilon)]-\frac{\partial}{\partial r}(\ln\mu)\left(\mathrm{i}\beta E_r-\frac{\partial E_z}{\partial r}\right)$$
$$=(-\omega^2\varepsilon\mu+\mathrm{i}\omega\mu\sigma)E_z. \tag{2.12}$$

Equations (2.11) and (2.12) constitute a set of coupled differential equations of second order for the functions E_r and E_z. Some of the properties of its solutions which we will need for the expansion theorem are given by the lemma below; i.e. the lemma shows the existence of solutions to (2.11) and (2.12) which are equivalent to the Jost function solutions, occurring in potential scattering theory.

We will formulate the completeness-expansion theorem for the modes of the fibre in a finite interval $0\leqslant r\leqslant a$, instead of the infinite interval $0\leqslant r<+\infty$, for mathematical convenience. The infinite case is then obtained by letting $a\to\infty$, and the appropriate completeness-expansion theorem for this case is stated in corollary 1, following the theorem.

Lemma. The set of coupled ordinary differential equations has solutions $E^{(1,2)}_{(r,z)}(r;\beta)$ in the interval $0\leqslant r\leqslant a$ such that

$$|E^{(1)}_{r,z}(r;\beta)|=\mathrm{O}\{\beta^{-1/2}\exp[-\mathrm{Im}(\beta)r]\},\qquad \text{if } \mathrm{Im}\,\beta\geqslant 0, \tag{2.13}$$

$$|E^{(2)}_{r,z}(r;\beta)|=\mathrm{O}\{\beta^{-1/2}\exp[\mathrm{Im}(\beta)r]\},\qquad \text{if } \mathrm{Im}\,\beta\leqslant 0. \tag{2.14}$$

The functions $E^{(1)}_{r,z}+E^{(2)}_{r,z}$ are entire functions of β, whereas the functions $E^{(1)}_{r,z}$ and $E^{(2)}_{r,z}$ are analytic in the complex β-plane with the exception of the point $\beta=0$, which is a branch point.

Proof. We rewrite (2.11) and (2.12) as a set of coupled integrodifferential equations:

$$E^{(1,2)}_r(r;\beta)=H^{(1,2)}_l(\beta r)+\int_r^a G(r,r';\beta)\left(-\frac{2}{r'}\frac{\partial}{\partial r'}[E_r(r')]\right.$$
$$-\frac{2}{r'^2}E_r(r')\frac{\partial}{\partial r'}\{\ln[\sigma(r')-\mathrm{i}\omega\varepsilon(r')]\}-\frac{2\mathrm{i}\beta}{r'}E_z(r')$$
$$-\frac{\partial}{\partial r'}[E_r(r')]\frac{\partial}{\partial r'}\{\ln[\sigma(r')-\mathrm{i}\omega\varepsilon(r')]\}$$
$$\left.+[\mathrm{i}\omega\mu(r')\sigma(r')-\omega^2\varepsilon(r')\mu(r')]E_r(r')\right)\,\mathrm{d}r', \tag{2.15a}$$

$$E_z^{(1,2)}(r;\beta)=H_l^{(1,2)}(\beta r)+\int_r^a G(r,r';\beta)\bigg[-\mathrm{i}\beta E_r(r')\frac{\partial}{\partial r'}\ln[\sigma(r')-\mathrm{i}\omega\varepsilon(r')]$$
$$+\left(\mathrm{i}\beta E_r(r')-\frac{\partial E_z(r')}{\partial r'}\right)\frac{\partial}{\partial r'}[\ln\mu(r')]$$
$$+[\mathrm{i}\omega\mu(r')\sigma(r')-\omega^2\varepsilon(r')\mu(r')]E_z(r')\bigg]\,\mathrm{d}r', \tag{2.15b}$$

where G denotes the Green function to Bessel's equation which is regular at the origin, zero if $r'\leqslant r$, and bounded at infinity for real values of β:

$$G(r,r';\beta)=0, \qquad \text{if } r'\leqslant r, \tag{2.16a}$$
$$=H_l^{(1)}(\beta r')H_l^{(2)}(\beta r)-H_l^{(2)}(\beta r')H_l^{(1)}(\beta r), \qquad \text{if } r'\geqslant r. \tag{2.16b}$$

The coupled set of integrodifferential equations (2.15a) and (2.15b) is easily rewritten as a set of coupled Volterra integral equations on integration by parts and observing that by virtue of (2.15a), (2.15b) and (2.16a),

$$\begin{aligned}&G(r,r;\beta)=0,\\&E_r^{(1,2)}(a;\beta)=H_l^{(1,2)}(\beta a), \qquad E_z^{(1,2)}(a;\beta)=H_l^{(1,2)}(\beta a).\end{aligned} \tag{2.17}$$

We therefore obtain

$$\int_r^a G(r,r';\beta)\left(\frac{1}{r'}\frac{\partial}{\partial r'}[E_r(r')]\right)\mathrm{d}r'$$
$$=\frac{1}{a}H_l^{(1,2)}(\beta a)G(r,a;\beta)-\int_r^a\frac{\partial}{\partial r'}\left(\frac{1}{r'}G(r,r';\beta)\right)E_r(r')\,\mathrm{d}r'$$
$$\times\int_r^a G(r,r';\beta)\frac{\partial}{\partial r'}(E_r(r'))\frac{\partial}{\partial r'}\{\ln[\sigma(r')-\mathrm{i}\omega\varepsilon(r')]\}$$
$$=H_l^{(1,2)}(\beta a)\frac{\partial}{\partial a}\{\ln[\sigma(a)-\mathrm{i}\omega\varepsilon(a)]\}G(r,a;\beta)$$
$$-\int_r^a\frac{\partial}{\partial r'}[G(r,r';\beta)]E_r(r')\frac{\partial}{\partial r'}\{\ln[\sigma(r')-\mathrm{i}\omega\varepsilon(r')]\}\,\mathrm{d}r', \tag{2.18}$$

$$\int_r^a G(r,r';\beta)\frac{\partial}{\partial r'}[E_z(r')]\frac{\partial}{\partial r'}[\ln\mu(r')]\,\mathrm{d}r'$$
$$=H_l^{(1,2)}(\beta a)G(r,a;\beta)\frac{\partial}{\partial a}[\ln\mu(a)]$$
$$-\int_r^a\frac{\partial}{\partial r'}\left(G(r,r';\beta)\frac{\partial}{\partial r'}[\ln\mu(r')]\right)E_z(r')\,\mathrm{d}r', \tag{2.19}$$

which, inserted into (2.15a) and (2.15b), leads to the desired set of coupled Volterra integral equations of the second kind.

The asymptotic behaviour of e.g. $E_r^{(1)}$ for large *positive* values of the imaginary part of β is deduced from the asymptotic expansions of the Hankel functions:

$$H_l^{(1,2)}(\beta r)=\left(\frac{2}{\pi\beta r}\right)^{1/2}\exp[\pm\mathrm{i}(\beta r-\tfrac{1}{2}l\pi-\tfrac{1}{4}\pi)]\left[1+\mathrm{O}\left(\frac{1}{\beta}\right)\right], \qquad 0\leqslant\arg\beta\leqslant2\pi. \tag{2.20}$$

Equation (2.16b) yields

$$G(r, r'; \beta) = \frac{2}{\pi}(rr')^{-1/2} \sin \beta(r - r')\left[1 + O\left(\frac{1}{\beta}\right)\right], \qquad 0 \leq \arg \beta \leq 2\pi. \tag{2.21}$$

We introduce two functions $e_r(r; \beta)$ and $e_z(r; \beta)$ such that

$$E_r^{(1)}(r; \beta) = H_l^{(1)}(r\beta)e_r(r; \beta), \qquad E_z^{(1)}(r; \beta) = H_l^{(1)}(r\beta)e_z(r; \beta). \tag{2.22}$$

If we insert (2.22) into (2.15a) and (2.15b), and use equations (2.18) and (2.19), we obviously end up with a coupled set of Volterra integral equations of the second kind for the functions e_r and e_z. However, the functions e_r and e_z are bounded functions for β if $|\beta| \to \infty$ and $0 \leq \arg \beta \leq \pi$:

$$|e_r(r; \beta)| < M, \qquad |e_z(r; \beta)| < M,$$

$$|\beta| \to \infty, \qquad 0 \leq \arg \beta \leq \pi, \qquad 0 \leq r \leq a, \tag{2.23}$$

where M denotes a positive number independent of r and β.

The property (2.23) is proven by inserting (2.22) into (2.15a) and (2.15b), using equations (2.18) and (2.19) and dividing both sides of the resulting equations by $H_l^{(1)}(r\beta)$. If we then insert the asymptotic expansion (2.20) with respect to β for the Hankel function of the first kind into the resulting set of equations, we observe that we end up with a set of coupled Volterra integral equations of the first kind for the functions e_r and e_z whose kernels either tend to zero if $|\beta| \to \infty$, $0 \leq \arg \beta \leq \pi$, or tend to a function of r and r' independent of β. The inhomogeneous term becomes 1. We consider as an example the term

$$\left(\frac{1}{r'}\frac{\partial}{\partial r'}[G(r, r'; \beta)]\right)E_r(r')[H_l^{(1)}(\beta r)]^{-1} = \psi(r, r'; \beta), \qquad 0 \leq r \leq r' \leq a. \tag{2.24}$$

Inserting (2.20) and (2.22) into (2.24) yields

$$\psi(r, r'\beta) = (2/\pi)e_r(r'; \beta)(r')^{-1.5}r^{-1/2}[1 + O(1/\beta)]. \tag{2.25}$$

The set of coupled Volterra equations for the functions $e_r(r; \beta)$ and $e_z(r; \beta)$ whose kernels tend to functions of r and r' but are independent of β can always be solved by a Liouville-Neumann series containing iterated kernels. This series converges absolutely and uniformly for large values of $|\beta|$, $0 \leq \arg \beta \leq \pi$, because there exists an absolutely converging comparison series

$$\sum_{n=0}^{\infty} \frac{M^n}{n!}, \tag{2.26}$$

where

$$M(\beta) = \int_0^a \int_r^a \sum_j |K_j(r, r'; \beta)| \, dr \, dr' \tag{2.27}$$

(see the appendix, equations (A6)-(A10)).

The summation is to be extended over all the kernels K_j occurring in the set of integral equations for e_r and e_z.

The kernels K_j tend either to zero or to a function independent of β so that the comparison series tends uniformly in β to a limit if $|\beta|$ tends to infinity, $0 \leq \arg \beta \leq \pi$. The Liouville-Neumann series for e_r and e_z therefore converges uniformly in β to a function independent of β, because a series in comparison with an absolutely and uniformly convergent series converges itself uniformly in a parameter like β, and limit and summation can be interchanged. We therefore showed that the Liouville-Neumann series for e_r and e_z tend to a function independent of β if $\beta \to \infty$, $0 \leqslant \arg \beta \leqslant \pi$ because we can interchange summation and $\lim_{|\beta| \to \infty}$ and can therefore take the values of the iterated kernels corresponding to $|\beta| = \infty$, $0 \leqslant \arg \beta \leqslant \pi$. This proves equation (2.23).

This completes the proof of equation (2.13). We similarly prove equation (2.14) introducing the functions $e_r^{(2)}$ and $e_z^{(2)}$ by

$$E_r^{(2)}(r;\beta) = H_l^{(2)}(r\beta)e_r^{(2)}(r;\beta), \qquad E_z^{(2)}(r;\beta) = H_l^{(2)}(r\beta)e_z^{(2)}(r;\beta). \tag{2.28}$$

We will now prove the analyticity of the functions $E_{r,z}^{(1,2)}(r;\beta)$ for all complex values of β, except $\beta = 0$. Inserting

$$H_l^{(1,2)}(r\beta) = J_l(r\beta) \pm \mathrm{i} N_l(r\beta), \tag{2.29}$$

$$\pi N_l(r\beta) = 2J_n(r\beta)\ln\left(\frac{r\beta}{2}\right) + C - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{r\beta}{2}\right)^{2k-n}$$
$$-\left(\frac{r\beta}{2}\right)^n \frac{1}{n}! \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{\infty} \frac{(-1)^k (r\beta/2)^{n+2k}}{k!(k+n)!}\left(\sum_{m=1}^{n+k} \frac{1}{m} + \sum_{m=1}^{k} \frac{1}{m}\right), \tag{2.30}$$

into (2.16*b*) shows that the Green function $G(r, r'; \beta)$ is an entire function of β because $J_l(r\beta)$ is an entire function of β for integer values of l. Equations (2.15*a*), (2.15*b*), (2.18) and (2.19) therefore show that the kernels of the integral equations are entire functions of β. This implies, however, that the resolvent kernel, $\boldsymbol{\Gamma}$ (appendix, equation (A7)), is also an entire function of β because, as we already remarked above, there exists an absolutely convergent comparison series which converges uniformly for all values of β belonging to any bounded domain D of the complex β plane. The uniform convergence follows from the fact that we obtain a comparison series $\forall \beta \in D$ taking $M = \{\max M(\beta) \colon \beta \in D\}$.

The solution of equations (2.15*a*) and (2.15*b*) can be written as (see the appendix)

$$\boldsymbol{E}^{(1,2)}(r;\beta) = \boldsymbol{H}_l^{(1,2)}(r\beta) + \int_r^a \boldsymbol{\Gamma}(r, r'; \beta)\boldsymbol{H}_l^{(1,2)}(r'\beta)\,\mathrm{d}r', \tag{2.31}$$

where the dyadic $\boldsymbol{\Gamma}$ is an entire function of β. Equation (2.29) therefore shows that $\boldsymbol{E}^{(1)} - \boldsymbol{E}^{(2)}$ is an entire function of β because the Bessel function $J_l(\beta r)$ is an entire function of β for integer values of l and

$$2J_l(\beta r) = H_l^{(1)}(\beta r) + H_l^{(2)}(\beta r). \tag{2.32}$$

Equation (2.31) also shows that the functions $\boldsymbol{E}^{(1,2)}(r;\beta)$ are analytic in the complex β plane with the exception of the point $\beta = 0$ where they have a logarithmic singularity because the Hankel functions of the first and second kind show this behaviour (see equations (2.29) and (2.30)).

We are now prepared to prove the core results of this paper, namely a class of expansion theorems for the field vectors E_r and E_z.

Theorem. Let $E_r(r;\beta)$ and $E_z(r;\beta)$ denote solutions to equations (2.11) and (2.12) in the interval $0 \leqslant r \leqslant a$ which are regular at the origin, for which we choose the functions $E_{r,z}^{(1)} + E_{r,z}^{(2)}$, defined by equations (2.15*a*), (2.15*b*), (2.18) and (2.19):

$$E_{r,z} = E_{r,z}^{(1)} + E_{r,z}^{(2)}. \tag{2.33}$$

Let $R_1(\beta;\, a)$ and $R_2(\beta;\, a)$ denote functions such that

$$\frac{R_1(\beta;\, a)E^{(1)}_{r,z}(r;\, \beta) - R_2(\beta;\, a)E^{(2)}_{r,z}(r;\, \beta)}{R_1(\beta;\, a) + R_2(\beta;\, a)} = C(r;\, \beta, a) \tag{2.34}$$

is a meromorphic function of β and†

$$R_{1,2}(\beta;\, a) = \mathrm{O}[\beta \exp(\pm \mathrm{i}\beta a)], \qquad |\beta| \to \infty, \qquad 0 \leqslant \arg \beta \leqslant 2\pi. \tag{2.35}$$

Suppose that the numbers β_j denote the zeros of $R_1 + R_2$:

$$R_1(\beta_j;\, a) + R_2(\beta_j;\, a) = 0, \tag{2.36}$$

which are all supposed to be simple. Then the following completeness relations hold true:

$$\sum_j \beta_j \frac{R_1(\beta_j;\, a)}{(\partial/\partial\beta_j)[R_1(\beta_j;\, a) + R_2(\beta_j;\, a)]} E_{r,z}(r;\, \beta_j)E_{r,z}(r';\, \beta_j) = r^{-1}\delta(r - r'), \tag{2.37}$$

where the summation is to be taken over all numbers β_j satisfying (2.36).

Proof. We consider the functions

$$\int_0^r f(r')E_{r,z}(r';\, \beta)\, \mathrm{d}r'\, C(r;\, \beta, a) = D_1(r;\, \beta, a) \tag{2.38}$$

and

$$\int_r^a f(r')C(r';\, \beta, a)\, \mathrm{d}r'\, E_{r,z}(r;\, \beta) = D_2(r;\, \beta, a), \tag{2.39}$$

where $f(r')$ denotes a function of bounded variation defined on the interval $0 \leqslant r' \leqslant a$, together with the integrals

$$I^{(n)}_{1,2}(r) = \frac{1}{2\pi\mathrm{i}} \oint_{|\beta| = c_n} \beta D_{1,2}(r;\, \beta, a)\, \mathrm{d}\beta. \tag{2.40}$$

The numbers c_n tend to infinity if $n \to \infty$ in such a way that the circle $|\beta| = c_n$ passes between two consecutive zeros β_j and β_{j+1}. The function $E_{r,z}C$ is a meromorphic function of β so that the theorem of residues leads to

$$I^{(n)}_1(r) = \sum_j \beta_j \frac{R_1(\beta_j;\, a)E_{r,z}(r;\, \beta_j)}{[R_1(\beta_j;\, a) + R_2(\beta_j;\, a)]'} \int_0^r f(r')E_{r,z}(r';\, \beta_j)\, \mathrm{d}r', \tag{2.41}$$

$$I^{(n)}_2(r) = \sum_j \beta_j \frac{R_1(\beta_j;\, a)E_{r,z}(r;\, \beta_j)}{[R_1(\beta_j;\, a) + R_2(\beta_j;\, a)]'} \int_r^a f(r')E_{r,z}(r';\, \beta_j)\, \mathrm{d}r'. \tag{2.42}$$

The prime denotes differentiation with respect to β_j and the summation is to be extended over all zeros β_j whose modulus is smaller than c_n. We will now evaluate the integrals (2.41) and (2.42) on the contour. Inserting the asymptotic expansions (2.13), (2.14) and (2.35) into (2.38) and (2.39) yields

$$D_1(r;\, \beta, a) = \frac{1}{\pi} \int_0^r (rr')^{-1/2}[\exp \mathrm{i}\beta\, \mathrm{sgn}(\mathrm{Im}\, \beta)(r - r')f(r')\, \mathrm{d}r'] \left[1 + \mathrm{O}\left(\frac{1}{\beta}\right)\right], \tag{2.43}$$

$$D_2(r;\, \beta, a) = \frac{1}{\pi} \int_r^a (rr')^{-1/2}[\exp \mathrm{i}\beta\, \mathrm{sgn}(\mathrm{Im}\, \beta)(r' - r)f(r')\, \mathrm{d}r'] \left[1 + \mathrm{O}\left(\frac{1}{\beta}\right)\right],$$
$$0 \leqslant \arg \beta \leqslant 2\pi. \tag{2.44}$$

†Note that a possible choice is provided in the second corollary.

The integrands of equations (2.43) and (2.44) give a non-negligible contribution only if $|\beta|\to\infty$ near the point $r'=r$ which can be estimated by integration by parts. The result is (Hoenders 1978, Titchmarsh 1970, § 4.9)

$$D_1(r;\beta,a)=\frac{f(r)}{\pi r\beta}\left[1+O\left(\frac{1}{\beta}\right)\right],\qquad 0\leqslant\arg\beta\leqslant 2\pi, \tag{2.45}$$

$$D_2(r;\beta,a)=\frac{f(r)}{\pi r\beta}\left[1+O\left(\frac{1}{\beta}\right)\right],\qquad 0\leqslant\arg\beta\leqslant 2\pi. \tag{2.46}$$

Inserting (2.45) and (2.46) into (2.41) and (2.42) yields

$$I_{1,2}^{(n)}(r)=(\pi r)^{-1}f(r)[1+O(1/c_n)]. \tag{2.47}$$

Combination of (2.41), (2.42) and (2.47) yields

$$\begin{aligned} I_1^{(n)}(r)+I_2^{(n)}(r) &\\ &=(\pi r)^{-1}f(r)[1+O(1/c_n)]\\ &=\sum_j \beta_j\frac{R_1(\beta_j;a)E_{r,z}(r;\beta_j)}{[R_1(\beta_j;a)+R_2(\beta_j;a)]'}\int_0^a f(r')E_{r,z}(r')\,dr', \end{aligned} \tag{2.48}$$

which proves equation (2.37) if $c_n\to\infty$ because this relation has to be valid for *all* functions $f(x)$ of bounded variation.

Corollary 1. The expansion (2.48) changes into

$$r^{-1}f(r)=\int d\rho(\beta)E_{r,z}(r;\beta)\int_0^\infty E_{r,z}(r')f(r')\,dr', \tag{2.49}$$

if a tends to infinity (Stone 1928, Titchmarsh 1970, ch 3 and references cited). The density function $\rho(\beta)$ can be determined from the limiting form of the factor

$$\beta_j\frac{R_1(\beta_j;a)}{[R_1(\beta_j;a)+R_2(\beta_j;a)]'}, \tag{2.50}$$

if $a\to\infty$.

The expansion (2.49) usually contains a continuous and discrete part, which in practice can usually be determined rather easily as one knows the asymptotic expansions of the functions involved if a tends to infinity.

Corollary 2. The conditions of the theorem are satisfied choosing for the functions R_1 and R_2 the functions $E_{r,z}^{(1)}(a;\beta)$, resp. $E_{r,z}^{(2)}(a;\beta)$. We then obtain

$$\sum_j \beta_j\frac{E_{r,z}^{(1)}(a;\beta_j)}{(\partial/\partial\beta_j)[E_{r,z}(a;\beta_j)]}E_{r,z}(r;\beta_j)E_{r,z}(r';\beta_j)=r^{-1}\delta(r-r'). \tag{2.51}$$

3. Discussion

The aim of this paper was to prove the often postulated completeness and orthogonality relations for the modes of a fibre with a radially dependent index of refraction. This leads to a completeness relation, namely equation (2.37), and expansion coefficients, namely equation (2.48), which differ from those derived from equation (1.1): the expansion coefficients occurring in equation (2.48) are *uncoupled* in the field components E_r and E_z whereas the orthogonality condition (1.1) leads to expansion coefficients which are *coupled* in the field components E_r and E_ϕ.

This difference is very interesting and probably very useful for the analysis of scattering and diffraction problems.

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Appendix. The solution of a coupled set of Volterra integral equations of the second kind

We consider the following set of coupled Volterra integral equations of the second kind:

$$f_j(x) = g_j(x) + \sum_{l=1}^{N} \int_x^a K_{j,l}(x, x') f_l(x')\,dx', \qquad 1 \leq j \leq N, \qquad 0 \leq x \leq a. \tag{A1}$$

Equation (A1) has been considered by Weatherburn (1915), who showed that the theory of equation (A1) is the same as the theory for a single equation of this type, if we only rewrite (A1) in vector notation. We will give a short survey of the method and refer for mathematical details to Weatherburn.

Let $\boldsymbol{i}_1, \boldsymbol{i}_2, \ldots, \boldsymbol{i}_N$ denote the unit vectors of a cartesian system, and suppose that the vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ and the dyadic $\mathbf{K}$ are defined by

$$\boldsymbol{f} = \sum_{j=1}^{N} \boldsymbol{i}_j f_j, \qquad \boldsymbol{g} = \sum_{j=1}^{N} \boldsymbol{i}_j g_j, \qquad \mathbf{K} = \sum_{j=1}^{N} \sum_{l=1}^{N} \boldsymbol{i}_j \boldsymbol{i}_l K_{j,l}. \tag{A2}$$

The dyadic product of $\mathbf{K}$ with a vector $\boldsymbol{f}$ is a vector $\boldsymbol{h}$ whose jth component is given by

$$(\boldsymbol{h})_j = (\mathbf{K} \cdot \boldsymbol{f})_j = \sum_{l=1}^{N} K_{j,l} f_l. \tag{A3}$$

We similarly define the (j, l) component of the dyadic product $\mathbf{K} \cdot \mathbf{L}$ by

$$(\mathbf{K} \cdot \mathbf{L})_{j,l} = \sum_{l'=1}^{N} K_{j,l'} L_{l',l}. \tag{A4}$$

Equation (A1) then reads as

$$\boldsymbol{f}(x) = \boldsymbol{g}(x) + \int_x^a \mathbf{K}(x, x') \cdot \boldsymbol{f}(x')\,dx'. \tag{A5}$$

This equation can be solved by iteration:

$$\boldsymbol{f}(x) = \boldsymbol{g}(x) + \int_x^a \boldsymbol{\Gamma}(x, x') \cdot \boldsymbol{g}(x')\,dx', \tag{A6}$$

if

$$\boldsymbol{\Gamma}(x, x') = \mathbf{K}(x, x') + \sum_{n=1}^{\infty} \mathbf{K}^{(n)}(x, x'), \tag{A7}$$

$$\mathbf{K}^{(n)}(x, x') = \int_x^a \mathbf{K}(x, x_1) \cdot \mathbf{K}^{(n-1)}(x_1, x')\,dx_1. \tag{A8}$$

The Liouville-Neumann series (A7) has a comparison, as in the scalar case, with an absolutely convergent series S:

$$S = \sum_{n=1}^{\infty} \frac{M^n}{n!}(x-a)^n, \tag{A9}$$

if

$$M = \max|\mathbf{K}(x, x')| \qquad b \leqslant x \leqslant x' \leqslant a \tag{A10}$$

so that the resolvent $\mathbf{\Gamma}$ exists for all values $b \leqslant x \leqslant x' \leqslant a$ and is represented by the Liouville-Neumann series (A7).

References

Birkhoff G D 1908 *Trans. Am. Math. Soc.* **9** 373-95

Hoenders B J 1978 *J. Phys. A: Math. Gen.* **11** 1815-32

Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill)

Stone M H 1928 *Math. Z.* **28** 654-76

Tamarkin J 1927 *Math. Z.* **27** 1-54

Tamir T 1979 *Integrated Optics, Topics in Applied Physics* vol 7 (Berlin: Springer)

Tamir T and Oliner A A 1963 *Proc. IEEE* **51** 317

Titchmarsh E C 1970 *Eigenfunction Expansions Associated with Second Order Differential Equations* vols 1 and 2 (Oxford: Oxford University Press)

Weatherburn C E 1915 *Q. Pure Appl. Math.* **46** 334-56